THE SHORT WAVE ASYMPTOTIC SOLUTION FOR THE DIFFRACTED FIELD DUE TO PLANE TRANSVERSE WAVES INCIDENT ON A SPHERE

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PMM Vol.29, № 6, 1965, pp. 1023-1034

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(Received July 22, 1965)

Problems connected with the propagation of waves in elastic media are more complicated than acoustic and electromagnetic problems inasmuch as there are two velocities of propagation in elastic media, and it becomes necessary to consider the interaction of the longitudinal and transverse waves in the boundary conditions. A large number of works have been devoted to these problems in recent years. By way of example the papers [1 to 15] may be mentioned.

In [1] problems of the diffraction of elastic waves by circular and elliptic cylinders are formulated for various boundary conditions. The solution is obtained in the form of infinite series in the suitable orthogonal functions. These series converge sufficiently rapidly only when the linear dimensions of the cross section are small compared to the wave length. In the collection of papers [2] formulations are given for several problems of wave propagation in elastic media having cylindrical or spherical interfaces. A method of investigation of the infinite series which represent the solutions is proposed, and problems connected with the substantiation of the proposed method are discussed.

A quite general method of solution of problems of scattering of high-frequency elastic waves by curvilinear objects is proposed in [3 and 4]. However, the solutions obtained by this method are valid (in accordance with Kirchhoff's principle) only in the "illuminated" region. Many foreigh publications [5 to 10] on the scattering of elastic waves by a cylinder or sphere deal essentially with the Rayleigh case ($k_i a \leqslant 1$, where a is a characteristic dimension of the body). One term or the first few terms of series in the appropriate orthogonal functions are taken into account.

The diffraction of step waves by a cylinder has ordinarily been considered either in the illuminated region [11] or for $t > 2a/c$ (c is a wave velocity) [12 and 13].

In [14 and 15] the importance of finding the short-wave asymptotic solution for cases of steady state is poined out, since from this solution one for a transient problem can be obtained using the formula for the inversion of the Laplace transform. In [15] this method is used to solve the problem of the diffraction of a compression wave due to an impulsive line source by a rigid cylinder. The solution is obtained in the vicinity of the wave front.

In the present paper, as in previous ones [16 and 17], the method of Watson [18] is used. This method was further developed and amplified by Fok

[19], and has been applied previously to acoustic and electromagnetic problems [18 to 22]. The problem of the diffraction of plane transverse waves by a sphere is solved by this method and a short-wave asymptotic solution is found for the displacements in the scattered waves in the various regions of the elastic space: the illuminated region, the shadow, and the penumbra.

1. Formulation of the problem. A plane transverse wave with displacements polarized in the x-direction

$$\mathbf{W}_0 = \mathbf{w}_0 \exp(-i\omega t) = \mathbf{x}_0 \exp[-i(\omega t + k_2 z)] \tag{1.1}$$

is incident on a sphere of radius a (Fig.1) in a homogeneous elastic space.

The total displacement field for steady-state waves satisfies Equation

$$\frac{1}{k_1^2}\nabla\nabla \mathbf{w} - \frac{1}{k_2^2}\nabla\times\nabla\times\mathbf{w} + \mathbf{w} = 0 \tag{1.2}$$

In order to solve Equation (1.2), we consider three solutions of the vector wave equation [23] (1.3)

$$\mathbf{l}_{mn}^{\mp} = \mathbf{r}_0 \frac{\partial}{\partial r} z_n(k_1 r) P_n^m(\mu) \frac{\cos}{\sin} m\varphi + \boldsymbol{\vartheta}_0 \frac{z_n(k_1 r)}{r} \frac{\partial}{\partial\vartheta} P_n^m(\mu) \frac{\cos}{\sin} m\varphi \mp$$
$$\mp \boldsymbol{\varphi}_0 \frac{m}{r\sin\vartheta} z_n(k_1 r) P_n^m(\mu) \frac{\sin}{\cos} m\varphi \qquad (\mu = \cos\vartheta)$$

$$\mathbf{m}_{mn}^{\mp} = \mp \boldsymbol{\vartheta}_0 \frac{m}{\sin\vartheta} z_n(k_2 r) P_n^m(\mu) \frac{\sin}{\cos} m\varphi - \boldsymbol{\varphi}_0 z_n(k_2 r) \frac{\partial}{\partial\vartheta} P_n^m(\mu) \frac{\cos}{\sin} m\varphi$$

$$\mathbf{n}_{mn}^{\mp} = \mathbf{r}_0 \frac{n(n+1)}{k_2 r} z_n(k_2 r) P_n^m(\mu) \frac{\cos}{\sin} m\varphi + \boldsymbol{\vartheta}_0 \frac{1}{k_2 r} \frac{\partial}{\partial r}[r z_n(k_2 r)] \times$$
$$\times \frac{\partial}{\partial\vartheta} P_n^m(\mu) \frac{\cos}{\sin} m\varphi \mp \boldsymbol{\varphi}_0 \frac{m}{k_2 r \sin\vartheta} \frac{\partial}{\partial r}[r z_n(k_2 r)] P_n^m(\mu) \frac{\sin}{\cos} m\varphi$$

where k_1 and k_2 are the wave numbers of the longitudinal and transverse waves, $\mathbf{r}_0$, $\boldsymbol{\vartheta}_0$ and $\boldsymbol{\varphi}_0$ are the unit vectors of the spherical coordinate system, $P_n^m(\mu)$ are the associated Legendre functions, $z_n(\xi)$ are spherical Bessel functions $j_n(\xi)$ or spherical Hankel functions $h_n(\xi)$. The superscripts minus and plus $(\mp)$ correspond to the minus and plus signs and also to the disposition of the functions $\cos m\varphi$ and $\sin m\varphi$ on the right-hand side of (1.3). These solutions (1.3) of the vector wave equation in spherical coordinates are obtained directly from the characteristic solution of the corresponding scalar equation by applying the gradient and curl operators [23]. Therefore, the functions $\mathbf{l}$ correspond to langitudinal waves, and $\mathbf{m}$ and $\mathbf{n}$ to the two types of transverse waves.

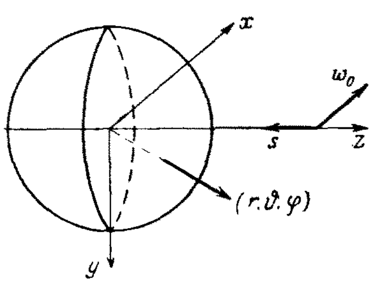

Fig. 1

We also expand the incident vector wave (1.1) into spherical wave functions [23]

$$\mathbf{w}_0 = \sum_{1}^{\infty} (-i)^n \frac{2n+1}{n(n+1)} [\mathbf{m}_{1n}^{+} + i\mathbf{n}_{1n}^{-}] \tag{1.4}$$

Then the scattered field due to the incident wave (1.4), satisfying Equation (1.2) outside the sphere can be written in the form [10]

$$\mathbf{w} = \sum_{1}^{\infty} (-i)^n \frac{2n+1}{n(n+1)} \left[\frac{a_n}{k_1} \mathbf{l}_{1n}^{(3)-} + \mathbf{m}_{1n}^{(1)+} + c_n \mathbf{m}_{1n}^{(2)+} + i\mathbf{n}_{1n}^{(1)-} + b_n \mathbf{n}_{1n}^{(2)-} \right] \quad (1.5)$$

The superscripts (1) and (2) refer to the functions $j_n(\xi)$ and $h_n^{(1)}(\xi)$ respectively; a_n, b_n and c_n are unknown coefficients which must be determined by the boundary conditions. The boundary conditions for $r = a$ are:

for the case of a perfectly rigid sphere of infinite density

$$u = v = w = 0 \quad (1.6)$$

for spherical surface which is free of traction

$$\sigma_{rr} = (\lambda + 2\mu) \frac{\partial u}{\partial r} + \lambda \left(\frac{2u}{r} + \frac{v}{r} \cot \vartheta + \frac{1}{r} \frac{\partial v}{\partial \theta} + \frac{1}{r \sin\theta} \frac{\partial w}{\partial \varphi} \right) = 0$$

$$\tau_{r\theta} = \mu \left(\frac{1}{r} \frac{\partial u}{\partial \theta} + \frac{\partial v}{\partial r} - \frac{v}{r} \right) = 0, \quad \tau_{r\varphi} = \mu \left(\frac{\partial w}{\partial r} - \frac{w}{r} + \frac{1}{r \sin\theta} \frac{\partial u}{\partial \varphi} \right) = 0 \quad (1.7)$$

where λ and μ are elastic constants. Using the boundary conditions (1.6) or (1.7) and the orthogonality relations for the wave functions $\mathbf{l}$, $\mathbf{m}$ and $\mathbf{n}$ [23], we obtain a system of equations for the determination of the coefficients in Equation (1.5):

for the case of the rigid sphere

$$a_n h_n^{(1)\prime}(x) + b_n \frac{n(n+1)}{y} h_n^{(1)}(y) = -i \frac{n(n+1)}{y} j_n(y), \quad c_n h_n^{(1)}(y) = -j_n(y)$$

$$\frac{a_n}{x} h_n^{(1)}(x) + \frac{b_n}{y} [h_n^{(1)}(y) + y h_n^{(1)\prime}(y)] = -\frac{i}{y} [j_n(y) + y j_n'(y)] \quad (1.8)$$

for the spherical cavity

$$a_n \left[2h_n^{(1)\prime}(x) - \frac{n(n-1) - \frac{1}{2}y^2}{x} h_n^{(1)}(x) \right] + b_n n(n+1) [y^{-1} h_n^{(1)}(y) - h_n^{(1)\prime}(y)] =$$
$$= -in(n+1) [y^{-1} j_n(y) - j_n'(y)]$$

$$a_n [h_n^{(1)\prime}(x) - x^{-1} h_n^{(1)}(x)] + b_n \left[\frac{n(n+1) - 1 - \frac{1}{2}y^2}{y} h_n^{(1)}(y) - h_n^{(1)\prime}(y) \right] =$$
$$= -i \left[\frac{n(n+1) - 1 - \frac{1}{2}y^2}{y} j_n(y) - j_n'(y) \right] \quad (1.9)$$

$$c_n [y h_n^{(1)\prime}(y) - h_n^{(1)}(y)] = -y j_n'(y) + j_n(y) \qquad (x = k_1 a, \; y = k_2 a)$$

Since the coefficent c_n of the function $\mathbf{m}$ in (1.5) occurs only in the last equations of the systems (1.8) and (1.9), the waves which are represented by $\mathbf{m}$ do not interact with the other two types of waves, those represented by the wave vectors $\mathbf{l}$ and $\mathbf{n}$ in (1.5). This shows that the wave vectors $\mathbf{m}$ describe the propagation of transverse SH-waves, polarized at right angles to the plane of incidence, and that $\mathbf{n}$ and $\mathbf{l}$ are transverse SV-waves polarized in the plane of incidence and longitudinal P-waves, respectively.

We introduce the superscript r for quantities which refer to the case of the rigid sphere and e for the free boundary, and we determine the coefficients in (1.5)

$$a_n = \frac{A_n}{\Delta}, \qquad b_n = -\frac{iB_n}{\Delta}, \qquad c_n = -\frac{C_n}{\Delta}$$

$$A_n^{\,r} = xy^{-2}n(n+1), \qquad A_n^{\,e} = 2xy^{-2}n(n+1)[2n(n+1)-4-y^2]$$

$$\Lambda^r = h_n^{(1)}(y), \qquad \Lambda^e = yh_n^{(1)'}(y) - h_n^{(1)}(y) \tag{1.10}$$

$$\Delta^r = xyh_n^{(1)'}(x)\,h_n^{(1)'}(y) + xh_n^{(1)'}(x)\,h_n^{(1)}(y) - n(n+1)\,h_n^{(1)}(x)\,h_n^{(1)}(y)$$

$$\Delta^e = 4[n(n+1)-2]\,xyh_n^{(1)'}(x)\,h_n^{(1)'}(y) - 2y^3h_n^{(1)}(x)\,h_n^{(1)'}(y) + \\ + 4[n(n+1)-y^2-2]\,xh_n^{(1)'}(x)\,h_n^{(1)}(y) - \\ - [4n^2(n+1)^2 + y^4 - 4y^2n(n+1) - 8n(n+1) + 2y^2]\,h_n^{(1)}(x)\,h_n^{(1)}(y)$$

The coefficients C_n and B_n are analogous to Λ and Δ, respectively, replacing $h_n^{(1)}(y)$ by $j_n(y)$. Thus, the displacement field assumes the form

$$\mathbf{w} = \sum_1^\infty (-i)^n \frac{2n+1}{n(n+1)}\Big\{\mathbf{r}_0 a_n h_n^{(1)'}(k_1r)\,P_n^1(\mu)\cos\varphi + \boldsymbol{\vartheta}_0 \frac{a_n}{k_1r}\,h_n^{(1)}(k_1r)\times \\ \times \frac{\partial P_n^1(\mu)}{\partial\theta}\cos\varphi - \boldsymbol{\varphi}_0 \frac{a_n}{k_1r}\,h_n^{(1)}(k_1r)\,\frac{P_n^1(\mu)}{\sin\vartheta}\sin\varphi + \\ + \mathbf{r}_0 \frac{n(n+1)}{k_2r}[b_nh_n^{(1)}(k_2r) + ij_n(k_2r)]\,P_n^1(\mu)\cos\varphi + \boldsymbol{\vartheta}_0 \frac{1}{k_2r}\frac{\partial}{\partial r}\times \\ \times [b_nrh_n^{(1)}(k_2r) + ij_n(k_2r)]\,\frac{\partial P_n^1(\mu)}{\partial\vartheta}\cos\varphi - \boldsymbol{\varphi}_0\frac{1}{k_2r}\frac{\partial}{\partial r}[b_nrh_n^{(1)}(k_2r) + ij_n(k_2r)]\times \\ \times\frac{P_n^1(\mu)}{\sin\vartheta}\sin\varphi + \boldsymbol{\vartheta}_0[c_nh_n^{(1)}(k_2r) + j_n(k_2r)]\,\frac{P_n^1(\mu)}{\sin\vartheta}\cos\varphi - \\ - \boldsymbol{\varphi}_0[c_nh_n^{(1)}(k_2r) + j_n(k_2r)]\,\frac{\partial P_n^1(\mu)}{\partial\vartheta}\sin\varphi\Big\} \tag{1.11}$$

It is known that the series representing the solution (1.11) converge. However, for high frequencies (k_1a and $k_2a \gg 1$), they converge very slowly and are not suitable for practical application.

2. The Watson transformation. In the study of the diffraction of high-frequency waves (k_1a and $k_2a \gg 1$) we shall make use of Watson's method [18] and we transform the series (1.11) into integrals in the complex plane ν (Fig.2).

$$\mathbf{w}^p = e^{-i\pi/4}\int_C \frac{\nu a_{\nu-1/2}}{\nu^2 - 1/4}\,\frac{\exp(-i\nu\pi/2)}{\cos\nu\pi}\,\mathbf{f}_1(\nu)\,d\nu$$

$$\mathbf{w}^{sv} = e^{-i\pi/4}\int_C \frac{\nu}{\nu^2 - 1/4}\,\frac{\exp(-i\nu\pi/2)}{\cos\nu\pi}\,[b_{\nu-1/2}\mathbf{f}_2(\nu) + i\mathbf{f}_2^\times(\nu)]\,d\nu \tag{2.1}$$

$$\mathbf{w}^{sh} = e^{-i\pi/4}\int_C \frac{|\nu}{\nu^2 - 1/4}\,\frac{\exp(-i\nu\pi/2)}{\cos\nu\pi}\,[c_{\nu-1/2}\mathbf{f}_3(\nu) + \mathbf{f}_3^\times(\nu)]\,d\nu$$

Here and in what follows

$$P_{\nu-1/2}^1(\mu) \equiv P_{\nu-1/2}^1, \quad P_{\nu-1/2}^1(-\mu) \equiv P_{\nu-1/2}^{1\times}$$

$$\mathbf{f}_1(\nu) = \mathbf{r}_0 h_{\nu-1/2}^{(1)'}(k_1r)\,P_{\nu-1/2}^{1\times}\cos\varphi + \boldsymbol{\vartheta}_0\frac{h_{\nu-1/2}^{(1)}(k_1r)}{k_1r}\,\frac{\partial P_{\nu-1/2}^{1\times}}{\partial\theta}\cos\varphi - \boldsymbol{\varphi}_0\frac{h_{\nu-1/2}^{(1)}(k_1r)}{k_1r}\,\frac{P_{\nu-1/2}^{1\times}}{\sin\vartheta}\sin\varphi$$

$$\mathbf{f}_2(\nu) = r_0 \frac{\nu^2 - {}^1/_4}{k_2 r} h^{(1)}_{\nu-{}^1/_2}(k_2 r) P^{1\times}_{\nu-{}^1/_2} \cos\varphi + \left[\vartheta_0 \frac{\partial}{\partial\theta} P^{1\times}_{\nu-{}^1/_2} \cos\varphi - \varphi_0 P^{1\times}_{\nu-{}^1/_2} \frac{\sin\varphi}{\sin\vartheta}\right] \times$$
$$\times \frac{1}{k_2 r} \frac{\partial}{\partial r} [r h^{(1)}_{\nu-{}^1/_2}(k_2 r)]$$
$$\mathbf{f}_3(\nu) = h^{(1)}_{\nu-{}^1/_2}(k_2 r) \left[\vartheta_0 P^{1\times}_{\nu-{}^1/_2} \frac{\cos\varphi}{\sin\vartheta} - \varphi_0 \frac{\partial}{\partial\theta} P^{1\times}_{\nu-{}^1/_2} \sin\varphi\right]$$

The expressions $\mathbf{f}_2^\times(\nu)$ and $\mathbf{f}_3^\times(\nu)$ differ from $\mathbf{f}_2(\nu)$ and $\mathbf{f}_3(\nu)$ in that the function $h^{(1)}_{\nu-{}^1/_2}(k_2 r)$ in the latter is replaced by $j_{\nu-{}^1/_2}(k_2 r)$.

The integrals of (2.1) a ong the contour C can be replaced by integrations along the path EF which encloses all the poles of the integrands of (2.1) as functions of ν which lie in the first quadrant. This is true because all the integrals along BD vanish by virtue of the oddness of the integrands and the integrals on the potions AB, DE and FG of the circle tend to zero as the radius of the circle approaches infinity

$$\mathbf{w}^p = e^{3i\pi/4} \int_E^F \frac{\nu}{\nu^2 - {}^1/_4} \frac{\exp(-i\nu\pi/2)}{\cos\nu\pi} \mathbf{f}_1(\nu) a_{\nu-{}^1/_2} d\nu$$

$$\mathbf{w}^{sv} = e^{3i\pi/4} \int_E^F \frac{\nu}{\nu^2 - {}^1/_4} \frac{\exp(-i\nu\pi/2)}{\cos\nu\pi} [b_{\nu-{}^1/_2} \mathbf{f}_2(\nu) + i\mathbf{f}_2^\times(\nu)] d\nu \tag{2.2}$$

$$\mathbf{w}^{sH} = e^{3i\pi/4} \int_E^F \frac{\nu}{\nu^2 - {}^1/_4} \frac{\exp(-i\nu\pi/2)}{\cos\nu\pi} [c_{\nu-{}^1/_2} \mathbf{f}_3(\nu) + \mathbf{f}_3^\times(\nu)] d\nu \tag{2.3}$$

We compute the integrals by means of the residues of the respective integrands at the poles ν_k and $\varkappa_k$

$$\mathbf{w}^p = -2\pi e^{i\pi/4} \sum_{\nu_k} \frac{\nu_k A_{\nu_k-{}^1/_2} \exp(-i\nu_k\pi/2)}{(\nu_k^2 - {}^1/_4)\cos\nu_k\pi\,(\partial\Delta/\partial\nu)_{\nu_k}} \mathbf{f}_1(\nu_k)$$

$$\mathbf{w}^{sv} = 2\pi e^{3i\pi/4} \sum_{\nu_k} \frac{\nu_k B_{\nu_k-{}^1/_2} \exp(-i\nu_k\pi/2)}{(\nu_k^2 - {}^1/_4)\cos\nu_k\pi(\partial\Delta/\partial\nu)_{\nu_k}} \mathbf{f}_2(\nu_k) \tag{2.4}$$

$$\mathbf{w}^{sH} = 2\pi e^{i\pi/4} \sum_{\varkappa_k} \frac{\varkappa_k C_{\varkappa_k-{}^1/_2} \exp(-i\varkappa_k\pi/2)}{(\varkappa_k^2 - {}^1/_4)\cos\varkappa_k\pi\,(\partial\Lambda/\partial\nu)_{\varkappa_k}} \mathbf{f}_3(\varkappa_k)$$

The convergence of these series follows from the convergence of the series of (1.11). All the equations given up to this point have been exact. Later we shall study a shoet-wave asymptotic solution for the displacements for $k_1 a$ and $k_2 a \gg 1$. Therefore, we shall replace the Bessel and Legendre functions contained in the solution by their asymptotic expressions [19 and 24].

The poles of the integrands of (2.3) are determined according to (1.10), by the zeros of the function Λ. Correct to terms of order $(k_1 a)^{-1}$ and $(k_2 a)^{-1}$

$$\Lambda^e \approx y h^{(1)\prime}_{\nu-{}^1/_2}(y) = 0, \qquad \Lambda^r \approx h^{(1)}_{\nu-{}^1/_2}(y) = 0 \tag{2.5}$$

Thus, $\varkappa_k{}^r$, the zeros of the functions Λ^r, coincide with the zeros of the spherical Hankel functions of the first kind and the $\varkappa_k{}^e$ with the roots of its derivative (2.5). The zeros of these functions are known [19 and 20]. In accordance with (1.10), the poles of the integrands in (2.2) coincide with the roots of Equations $\Delta = 0$, which assume the form

$$\Delta^e = 4\nu^2 xy h^{(1)\prime}_{\nu-{}^1/_2}(x) h^{(1)\prime}_{\nu-{}^1/_2}(y) - (2\nu^2 - y^2)^2 h^{(1)}_{\nu-{}^1/_2}(x) h^{(1)}_{\nu-{}^1/_2}(y) = 0$$
$$\Delta^r = xy h^{(1)\prime}_{\nu-{}^1/_2}(x) h^{(1)\prime}_{\nu-{}^1/_2}(y) - \nu^2 h^{(1)}_{\nu-{}^1/_2}(x) h^{(1)}_{\nu-{}^1/_2}(y) = 0 \tag{2.6}$$

for $k_1 a$ and $k_2 a \gg 1$.

Equations (2.6) coincide with the analogous ones in problems of diffraction by a cylinder [15 to 17], with the accuracy to constants. These equations have two series of roots located in the first quadrant which are approximately of the form

$$\lambda_k = x + ({}^1/_2 x)^{1/3} t_k, \qquad \mu_k = y + ({}^1/_2 y)^{1/3} \tau_k$$

(the Hankel-Fok approximation), where t_k and τ_k are the roots of Equations

$$w'(t) - q_1(t)\, w(t) = 0$$
$$w'(\tau) - q_2(\tau)\, w(\tau) = 0$$

Here $w(\xi) = u(\xi) + iv(\xi)$ is the Airy function [19]; for the poles which lie near x and y, respectively, $q_1(t)$ and $q_2(\tau)$ may be taken as constants

$$q_1^r = \frac{i\varepsilon}{\sqrt{1-\varepsilon^2}}\left(\frac{x}{2}\right)^{1/3}$$
$$q_1^e = \frac{i(2\varepsilon^2-1)^2}{4\varepsilon^3\sqrt{1-\varepsilon^2}}\left(\frac{x}{2}\right)^{1/3}$$
$$q_2^r = \frac{1}{\sqrt{1-\varepsilon^2}}\left(\frac{y}{2}\right)^{1/3} \qquad \left(\varepsilon = \frac{k_1}{k_2} < 1\right)$$
$$q_2^e = \frac{1}{4\sqrt{1-\varepsilon^2}}\left(\frac{y}{2}\right)^{1/3}$$

For the case of a cavity, the equation $\Delta^* = 0$ has one more real root $\nu^* = \varkappa y$ ($\varkappa > 1$). The residues of the integrands of (2.1) at this pole give the displacements in the Rayleigh surface wave [17].

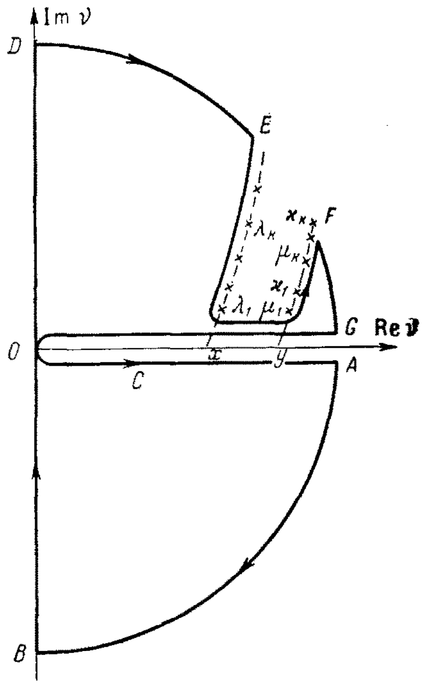

Fig. 2

3. Diffraction in the region of the shadow. Let us investigate the short-wave asymptotic solution for the displacements (2.4). We first estimate the order of magnitude of the displacement components in the longitudinal and transverse elastic waves for $k_1 a \gg 1$ using the asymptotic expressions for the spherical Bessel functions and the associated Legendre functions. It then turns out that the displacement components w^p, w^{sv}, v^{sh} in (1.11) are of order $(k_1 a)^{-1}$ compared to the remaining terms and may be neglected. With the aid of the asymptotic formulas [24]

$$P^{1\varkappa}_{\nu-1/2} = \frac{dP^{\varkappa}_{\nu-1/2}}{d\vartheta} = \left(\frac{2\nu}{\pi\sin\vartheta}\right)^{1/2}\sin\left[\nu(\pi-\vartheta) - {}^1/_4\pi\right] \qquad \text{for} \begin{cases} |\nu\sin\vartheta| \gg 1 \\ \operatorname{Re}\nu > {}^1/_2 \end{cases} \tag{3.1}$$
$$\frac{dP^{1\varkappa}_{\nu-1/2}}{d\vartheta} = \frac{d^2P^{\varkappa}_{\nu-1/2}}{d\vartheta^2} = -\left(\frac{2\nu^3}{\pi\sin\vartheta}\right)^{1/2}\cos\left[\nu(\pi-\vartheta) - {}^1/_4\pi\right]$$

and suitable asymptotic expressions for the spherical Bessel and Hankel functions [19], we obtain

$$\mathbf{w}_1{}^p = \frac{2\varepsilon^2 K \cos\varphi}{(1 - a^2/r^2)^{1/4}(1-\varepsilon^2)^{1/4}} \left(\frac{\pi}{\sin\vartheta}\right)^{1/2} e^{i\,[k_1\sqrt{r^2-a^2} - k_2 a\sqrt{1-\varepsilon^2} + 1/4\,\pi]}\, \frac{a}{r} \times$$

$$\times \sum_k \frac{\exp\,[i\lambda_k(\cos^{-1}\varepsilon - \cos^{-1} a/r - 1/2\,\pi)]}{w(t_k)(t_k - q_1{}^2)\cos\lambda_k\pi} \times$$

$$\times \left\{\mathbf{r}_0 i \left(1 - \frac{a^2}{r^2}\right)^{1/2} \sin\left[\lambda_k(\pi - \vartheta) - \frac{\pi}{4}\right] - \boldsymbol{\vartheta}_0 \frac{a}{r} \cos\left[\lambda_k(\pi-\vartheta) - \frac{\pi}{4}\right]\right\}$$

$$\left(K^r = 1, \quad K^e = 1 - \frac{1}{2\varepsilon^2}\right) \tag{3.2}$$

$$\mathbf{w}_{11}{}^{sv} = \frac{2\varepsilon^2 K^2 \cos\varphi}{(1 - \varepsilon^2 a^2/r^2)^{1/4}(1-\varepsilon^2)^{1/2}} \left(\frac{\pi}{\sin\vartheta}\right)^{1/2} e^{ik_2\left[\sqrt{r^2-a^2\varepsilon^2} - 2a\sqrt{1-\varepsilon^2}\right]} \left(\frac{k_1 a}{2}\right)^{1/6} \frac{a}{r} \times$$

$$\times \sum_k \frac{\exp\,[i\lambda_k(2\cos^{-1}\varepsilon - \cos^{-1}\varepsilon a/r - 1/2\,\pi)]}{(t_k - q_1{}^2)\cos\lambda_k\pi} \times$$

$$\times \left\{\mathbf{r}_0 \frac{\varepsilon a}{r} \sin\left[\lambda_k(\pi-\vartheta) - \frac{\pi}{4}\right] - \boldsymbol{\vartheta}_0 i \left(1 - \frac{\varepsilon^2 a^2}{r^2}\right)^{1/2} \cos\left[\lambda_k(\pi-\vartheta) - \frac{\pi}{4}\right]\right\} \tag{3.3}$$

$$\mathbf{w}_{12}{}^{sv} = \frac{2\cos\varphi}{(1 - a^2/r^2)^{1/4}} \left(\frac{\pi}{\sin\vartheta}\right)^{1/2} e^{ik_2\sqrt{r^2-a^2}} \left(\frac{k_2 a}{2}\right)^{-1/6} \frac{a}{r} \times$$

$$\times \frac{\exp\,[-i\mu_k(\cos^{-1} a/r + 1/2\,\pi)]}{w^2(\tau_k)(\tau_k - q_2{}^2)\cos\mu_k\pi} \left\{\mathbf{r}_0 \frac{a}{r} \sin\left[\mu_k(\pi-\vartheta) - \frac{\pi}{4}\right] -\right.$$

$$\left. - \boldsymbol{\vartheta}_0 i \left(1 - \frac{a^2}{r^2}\right)^{1/2} \cos\left[\mu_k(\pi-\vartheta) - \frac{\pi}{4}\right]\right\} \tag{3.4}$$

$$\mathbf{w}_1{}^{sh} = \frac{2i\varphi_0 \sin\varphi}{(1 - a^2/r^2)^{1/4}} \left(\frac{\pi}{\sin\vartheta}\right)^{1/2} e^{ik_2\sqrt{r^2-a^2}} \left(\frac{k_2 a}{2}\right)^{-1/6} \times$$

$$\times \sum_k I_{\varkappa_k} \exp\left[-i\varkappa_k\left(\cos^{-1} s\,\frac{a}{r} + \frac{\pi}{2}\right)\right] \frac{\cos\,[\varkappa_k(\pi-\vartheta) - 1/4\,\pi]}{\cos\varkappa_k\pi} \tag{3.5}$$

$$I_{\varkappa_k}{}^r = \frac{1}{[w'(\tau_k{}^0)]^2}, \qquad I_{\varkappa_k}{}^e = -\frac{1}{\tau_k' w^2(\tau_k')}, \qquad \varkappa_k = y + (1/2\,y)^{1/3}\tau_k$$

where $\tau_k{}^\circ$ are the roots of $w(\tau_k{}^\circ) = 0$, and τ_k' are the roots of $w'(\tau_k') = 0$. We shall now determine the regions in which the series converge rapidly.

The quantities λ_k, μ_k and $\varkappa_k$ have positive imaginary parts which increase as the number k becomes larger. Therefore, the series (3.2) to (3.5), converge quite rapidly if the following inequalities are respectively satisfied

$$\cos^{-1}\varepsilon - \cos^{-1}(a/r) + \vartheta - 1/2\pi > 0$$

$$2\cos^{-1}\varepsilon - \cos^{-1}(\varepsilon a/r) + \vartheta - 1/2\pi > 0$$

$$-\cos^{-1}(a/r) + \vartheta - 1/2\pi > 0$$

$$-\cos^{-1}(a/r) + \vartheta - 1/2\pi > 0$$

Thus, the series (3.2) for the londitudinal displacements will converge rapidly everywhere in the region of the shadow for longitudinal displacements, the boundary of which

$$r_1 = a / \cos(\vartheta - \alpha^\times)$$

is the truncated cone with vertex angle $\pi - 2\alpha^\times$ having as a directrix the parallel circle defined by $\alpha^{\times}$ ($\alpha^{\times}$ is the angle of total internal reflection).

The series (3.4) and (3.5) for the transverse displacements have a common region of convergence corresponding to the region of the geometric shadow; its boundary

$$r_2 = a / \cos(\vartheta - {}^1/_2\pi)$$

is the half-cylinder surface to the left of the equatorial circle (Fig.3).

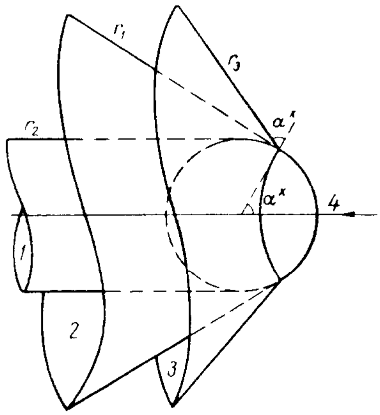

Fig. 3

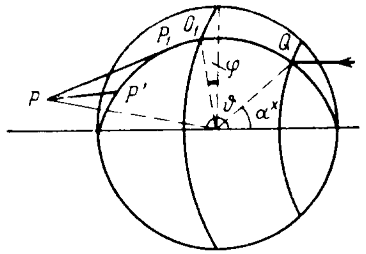

Fig. 4

The series (3.3) for transverse displacements describes the displacements in the diffracted waves of the head-wave type [17]. The boundary of its region of convergence (Fig.3)

$$r_3 = a\varepsilon / \cos(\vartheta + {}^1/_2\pi - 2\alpha^{\times})$$

is the surface of the truncated cone with vertex angle $2(\pi - 2\alpha^{\times})$ (the vertex angle for r_3 is twice as large as for r_1) and having the same α parallel circle as directrix.

The physical meaning of Equations (3.2) to (3.5) is as follows: they give the displacements in the diffracted waves which envelop the sphere.

The longitudinal diffracted waves (3.2) and the transverse ones of head-wave type (3.3) originate on the $\alpha^{\times}$-parallel circle where the incident transverse rays make the angle $\alpha^{\times}$ with the normal to the sphere. Here are formed the longitudinal diffracted waves which move on the surface of the sphere along the meridians emitting longitudinal waves tangentially and transverse waves at the angle $\alpha^{\times}$. As is easy to see from Expressions (1.11) for the displacements, only the SV polarized transverse waves take part in the formation of the longitudinal scattered waves (Fig.4). The incident transverse rays which are tangent to the sphere at its equator give rise to diffracted transverse SV and SH waves (3.4) and (3.5) which also propagate on the sphere along the meridians and are then emitted to the point of observation along the tangent (Fig.4). The presence of $\cos \nu\pi$ in the numerators of (3.2) to (3.5) shows that the diffracted waves which encircle the sphere 1, 2, ..., n... times before falling on the point of observation are also taken into account. The singularity $(\sin\vartheta)^{-1/2}$ indicates that the diffracted waves focus at the points of the axis $\vartheta = \pi$. Near $\vartheta = \pi$ Equations (3.1) are not valid (the condition $|\nu \sin\vartheta| \gg 1$ is not satisfied), and we should use the asymptotic expressions obtained from

$$P_{\nu^{\,1/_2}}(-\mu) \rightarrow I_0[\nu(\pi - \vartheta)]$$

valid for $|\nu| \gg 1$ and the quantity $|\nu(\pi - \vartheta)|$ comparable to unity.

Thus, for the scattering of elastic waves, unlike the acoustic and electromagnetic cases, there exist four types of different diffracted waves having different illuminated regions and shadows, as may be seen from Equations (3.2) to (3.5) (Fig.3).

4. Diffraction in the "illuminated" regions. In the regions where Equations (3.2) to (3.5) cease to hold (these are the illuminated regions for the corresponding types of waves), we shall proceed from Equations (2.2) and (2.3) in which we replace $P_{\nu-1/2}^{1\times}$ and $\partial P_{\nu-1/2}^{1\times}/\partial\vartheta$ by corresponding expressions which follow from (3.1) and

$$P_{\nu-1/2}^{\times} = \exp\left[i\left(\nu - {}^1/_2\right)\pi\right] P_{\nu-1/2} + 2i\cos\nu\pi\, Q_{\nu-1/2} \qquad (4.1)$$

We shall now compute the first terms of (2.2) and (2.3) which correspond to the first term of the right-hand side of (4.1). From the residues at the poles of the integrands we obtain

$$\mathbf{w}_1{}^p = \frac{2i\varepsilon^2 K\cos\varphi}{(1-a^2/r^2)^{1/4}(1-\varepsilon^2)^{1/4}}\left(\frac{\pi}{\sin\vartheta}\right)^{1/2} e^{i\left[k_1\sqrt{r^2-a^2}-k_2 a\sqrt{1-\varepsilon^2}+{}^1/_4\pi\right]}\frac{a}{r}\times$$
$$\times\sum_k \frac{\exp\left[i\lambda_k\left(\cos^{-1}\varepsilon - \cos^{-1}a/r + {}^1/_2\pi\right)\right]}{w(t_k)(t_k-q_1{}^2)\cos\lambda_k\pi}\left[\mathbf{r}_0 i\left(1-\frac{a^2}{r^2}\right)^{1/2}\sin\left(\lambda_k\vartheta-\frac{\pi}{4}\right)+\right.$$
$$\left.+\,\boldsymbol{\vartheta}_0\frac{a}{r}\cos\left(\lambda_k\vartheta-\frac{\pi}{4}\right)\right]+\mathbf{w}_g{}^p \equiv \mathbf{w}_2{}^p+\mathbf{w}_g{}^p \qquad (4.2)$$

$$\mathbf{w}^{sv} \equiv \mathbf{w}_{21}{}^{sv}+\mathbf{w}_{22}{}^{sv}+\mathbf{w}_g{}^{sv}$$

$$\mathbf{w}_{21}{}^{sv} = \frac{2i\varepsilon^2 K^2\cos\varphi K}{(1-\varepsilon^2a^2/r^2)^{1/4}(1-\varepsilon^2)^{1/2}}\left(\frac{\pi}{\sin\vartheta}\right)^{1/2} e^{ik_2\left[\sqrt{r^2-a^2\varepsilon^2}-2a\sqrt{1-\varepsilon^2}\right]}\left(\frac{k_1 a}{2}\right)^{1/4}\frac{a}{r}\times$$
$$\times\sum_k\frac{\exp\left[i\lambda_k\left(2\cos^{-1}\varepsilon - \cos^{-1}\varepsilon a/r + {}^1/_2\pi\right)\right]}{(t_k-q_1{}^2)\cos\lambda_k\pi}\times$$
$$\times\left[\mathbf{r}_0\frac{\varepsilon a}{r}\sin\left(\lambda_k\vartheta-\frac{\pi}{4}\right)+\boldsymbol{\vartheta}_0 i\left(1-\frac{\varepsilon^2a^2}{r^2}\right)^{1/2}\cos\left(\lambda_k\vartheta-\frac{\pi}{4}\right)\right] \qquad (4.3)$$

$$\mathbf{w}_{22}{}^{sv} = \frac{2i\cos\varphi}{(1-a^2/r^2)^{1/4}}\left(\frac{\pi}{\sin\vartheta}\right)^{1/2} e^{ik_2\sqrt{r^2-a^2}}\left(\frac{k_2 a}{2}\right)^{-1/6}\frac{a}{r}\times$$
$$\times\sum_k\frac{\exp\left[-i\mu_k\left(\cos^{-1}a/r-{}^1/_2\pi\right)\right]}{w^2(\tau_k)(\tau_k-q_2{}^2)\cos\mu_k\pi}\left[\mathbf{r}_0\frac{a}{r}\sin\left(\mu_k\vartheta-\frac{\pi}{4}\right)+\right.$$
$$\left.+\,\boldsymbol{\vartheta}_0 i\left(1-\frac{a^2}{r^2}\right)^{1/2}\cos\left(\mu_k\vartheta-\frac{\pi}{4}\right)\right] \qquad (4.4)$$

$$\mathbf{w}_1{}^{sh} = \frac{2\varphi_0\sin\varphi}{(1-a^2/r^2)^{1/4}}\left(\frac{\pi}{\sin\vartheta}\right)^{1/2} e^{ik_2\sqrt{r^2-a^2}}\left(\frac{k_2a}{2}\right)^{-1/6}\frac{a}{r}\times$$
$$\times\sum_k I_{\varkappa_k}\frac{\exp\left[-i\varkappa_k\left(\cos^{-1}a/r-{}^1/_2\pi\right)\right]}{\cos\varkappa_k\pi}\cos\left(\varkappa_k\vartheta-{}^1/_4\pi\right)\equiv\mathbf{w}_2{}^{sh}+\mathbf{w}_g{}^{sh} \qquad (4.5)$$

where the subscript g denotes those terms in (2.2) and (2.3) which correspond to the second term on the right of (4.1). Using the Debye asymptotic approximations for the spherical Bessel and Hankel functions and the asymptotic expression

$$Q_{\nu-1/2} = \frac{\exp i\left(\nu\vartheta-{}^1/_4\pi\right)}{\sqrt{2\pi\nu\sin\vartheta}} \qquad \left(|\nu\sin\vartheta|\gg 1,\ \operatorname{Re}\nu > {}^3/_2\right)$$

we compute $\mathbf{w}_g{}^p$, $\mathbf{w}_g{}^{sv}$ and $\mathbf{w}_g{}^{sh}$ by the method of steepest descents. We construct the path of steepest descent as in [17] and obtain

$$\mathbf{w}_g{}^p = 2\cos\varphi \frac{a}{r}\left(\frac{\sin\alpha_1}{\sin\vartheta}\right)^{1/2} \frac{\Omega(\varepsilon, a, \alpha_1)\sin\alpha_1\cos\alpha_1\, U}{W(\varepsilon, \alpha_1)\, D_+(\alpha_1)}\left[\mathbf{r}_0\Omega(\varepsilon, r, \alpha_1) + \right.$$
$$\left. + \boldsymbol{\vartheta}_0 \frac{a}{r}\sin\alpha_1\right]\exp\{ik_2 a\,[\Omega(\varepsilon, r, \alpha_1) - \Omega(\varepsilon, a, \alpha_1) - \cos\alpha_1]\} \qquad (4.6)$$

$$\mathbf{w}_g{}^{sv} = \cos\varphi \frac{a}{r}\left(\frac{\sin\alpha_2}{\sin\vartheta}\right)^{1/2} \frac{\cos\alpha_2\, D_-(\alpha_2)}{W(1, \alpha_2)\, D_+(\alpha_2)}\left[\mathbf{r}_0 \frac{a}{r}\sin\alpha_2 - \right.$$
$$\left. - \boldsymbol{\vartheta}_0\Omega(1, r, \alpha_2)\right]\exp\left\{ik_2 r\left[\Omega(1, r, \alpha_2) - 2\frac{a}{r}\cos\alpha_2\right]\right\} + \mathbf{w}_0{}^{sv} \qquad (4.7)$$

$$\mathbf{w}_g{}^{sh} = \mp\sin\varphi \frac{a}{r}\left(\frac{\sin\alpha_2}{\sin\vartheta}\right)^{1/2} \frac{\cos\alpha_2}{W(1, \alpha_2)}\boldsymbol{\varphi}_0\exp\left\{ik_2 r\left[\Omega(1, r, \alpha_2) - \right.\right.$$
$$\left.\left. - 2\frac{a}{r}\cos\alpha_2\right]\right\} + \mathbf{w}_0{}^{sh} \qquad (4.8)$$

where the $\mathbf{w}_0$ are the displacements of the incident wave

$$U^r = 1, \quad U^e = 2(2\sin^2\alpha_1 - 1), \quad \Omega(\varepsilon, r, \alpha) = \sqrt{\varepsilon^2 - a^2\sin^2\alpha / r^2}$$
$$W(\varepsilon, \alpha) = \sqrt{\Omega(\varepsilon, r, \alpha)[\Omega(\varepsilon, a, \alpha) + \cos\alpha] - (a/r)\cos\alpha\Omega(\varepsilon, a, \alpha)}$$
$$D_\pm{}^e(\alpha) = (2\sin^2\alpha - 1)^2 \pm 4\sin^2\alpha\cos\alpha\Omega(\varepsilon, a, \alpha)$$
$$D_\pm{}^r(\alpha) = \sin^2\alpha \pm \cos\alpha\Omega(\varepsilon, a, \alpha)$$

The geometric significance of the angles α_1 and α_2 are indicated in Fig.5. They are the angles of incidence of the transverse rays which are reflected as those longitudinal and transverse rays arriving at the point of observation $P(r, \vartheta, \varphi)$.

In Equation (4.8) the upper sign refers to the case of the rigid sphere, and the lower one to the cavity. It is easy to see that Equations (4.6) to (4.8) give the displacements in the waves which are reflected from the sphere in accordance with the rules of the geometric theory. Equations (4.7) and (4.8) are valid in the illuminated region for transverse displacements (Fig.3) and cease to be correct in the region of the penumbra ω_2 near the boundary Γ_2 of the geometric shadow. In the region ω_2, we have $|\nu^2 - y^2| \sim Ay^{4/3}$ ($A \sim 2.5$) and the Debye asymptotic expression for the function $h^{(1),(2)}_{\nu-1/2}(y)$ which is applied in the derivation of (4.7) and (4.8) is not suitable there. Furthermore, (4.7) is also invalid in the transition region ω_3 near the surface of the cone Γ_3, since there the Debye asymptotic expression for the function $h^{(1)}_{\nu-1/2}(x)$. is not correct. Equation (4.6) for the longitudinal displacements is valid in the illuminated region for longitudinal waves and violates in its penumbra ω_1 (Fig.3) near the cone Γ_1. It should be noted that in the deformation of the path EF into a path of steepest descent, poles of the function $b_{\nu-1/2}$ sometimes happen to fall between the two paths. These poles are taken into account in the same way as in [17] in obtaining the final expressions for the displacements.

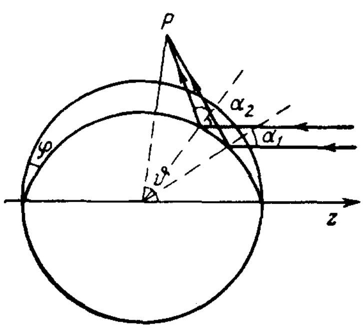

Fig. 5

5. Calculation of the displacements in the transition regions. In the regions ω_1, ω_2 and ω_3 the displacements are found in accordance with Equations (4.2) to (4.5). However, the terms corresonding to $\mathbf{w}_g{}^p$, $\mathbf{w}_g{}^{sv}$ and $\mathbf{w}_g{}^{sh}$ should be computed, following Fok [19], by quadratures using the more complicated Hankel-Fok asymptotic expressions for $h^{(1)}_{\nu-1/2}(x)$ and $h^{(1),(2)}_{\nu-1/2}(y)$.

In this connection, it is convenient to transform the contour of integration EF into the path Γ (Fig.6) in the plane

$$t = (\nu - x)\,(^1/_2 x)^{1/3}$$

for the region ω_1 and ω_3 and the analogous path Γ' in the plane

$$\tau = (\nu - y)\,(^1/_2 y)^{1/3}$$

for the region ω_2. As a result, we obtain

$$\mathbf{w}_g{}^{\nu} = -\frac{a}{r}\frac{\varepsilon^2\cos\varphi}{\sqrt{\pi\sin\vartheta}}\int\limits_{\Gamma}\frac{K(t)\,p_1{}^{3/2}(t)\,[\mathbf{r}_0V^2(t,\,r,\,1)+\vartheta_0\,(a/r)\,p_1(t)]}{V(t,\,r,\,1)\,V(t,\,a,\,\varepsilon)\,[w'(t)-q_1(t)\,w(t)]}\,e^{i\psi_1(t)}\,dt \qquad (5.1)$$

$$\psi_1(t) = xp_1(t)\,[\vartheta - {}^1/_2\pi - \cos^{-1}(a/r)\,p_1(t) + \cos^{-1}\varepsilon p_1(t)] +$$
$$+ k_1rV^2(t,\,r,\,1) - k_2aV^2(t,\,a,\,\varepsilon)$$

$$V(t,\,r,\,\varepsilon) = \left(1 - \frac{\varepsilon^2a^2p_1{}^2(t)}{r^2}\right)^{1/4}, \qquad K^r(t) = 1, \qquad K^e(t) = 1 - \frac{1}{2\varepsilon^2p_1{}^2(t)}$$

$$q_1{}^r(t) = \frac{ip_1{}^2(t)\,\varepsilon}{V^2(t,\,a,\,\varepsilon)}\left(\frac{k_1a}{2}\right)^{1/3}, \qquad q_1{}^e(t) = i\,\frac{[2p_1{}^2(t)\,\varepsilon^2-1]^2}{4\varepsilon^3p_1{}^2(t)\,V^2(t,\,a,\,\varepsilon)}\left(\frac{k_1a}{2}\right)^{1/3}$$

Further, the integral in (5.1) should be split into two integrals, one from 0 to ∞ along the real axis and the other from $\infty\exp(^2/_3\pi t)$ to 0 along $\arc t = {}^2/_3\pi$. Using the formula for the Airy function on the ray $\arc t = {}^2/_3\pi$, we can also reduce the last integral to one with real limits

$$\mathbf{w}_g{}^{\nu} = -\frac{a}{r}\frac{\varepsilon^2\cos\varphi}{\sqrt{\pi\sin\vartheta}}\left(\int\limits_0^{\infty}\frac{K(t)\,p_1{}^{3/2}(t)\,[\mathbf{r}_0V^2(t,\,r,\,1)+\vartheta_0\,(a/r)\,p_1(t)]}{V(t,\,r,\,1)\,V(t,\,a,\,\varepsilon)\,[w'(t)-q_1(t)\,w(t)]}\,e^{i\psi_1(t)}\,dt + \right.$$

$$\left. + \int\limits_0^{\infty}\frac{K(te^{2/3\pi i})\,\{\mathbf{r}_0V^2(te^{2/3\pi i},\,r,\,1)+\vartheta_0\,(a/r)\,p_1(te^{2/3\pi i})\}\,p_1{}^{3/2}(te^{2/3\pi i})\exp[i\psi_1(te^{2/3\pi i})]}{V(te^{2/3\pi i},\,r,\,1)\,V(te^{2/3\pi i},\,a,\varepsilon)\,[\overline{w}'(t)-e^{2/3\pi i}q_1(te^{2/3\pi i})\,\overline{w}(t)]}\,dt\right) \qquad (5.2)$$

$$\left(p_1(t) = 1 + \left(\frac{x}{2}\right)^{-2/3}\frac{t}{2}\right)$$

The integrals of (5.2) are calculated by the method of numerical quadrature with the aid of tables of the functions $u(t)$ and $v(t)$ for real, positive t, which are given in [19].

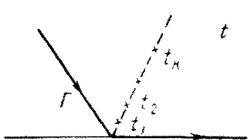

Fig. 6

For large negative values of t, i.e. in the illuminated region, the integral (5.1) is computed by the method of steepest descents with the aid of the asymptotic expressions [19]

$$w(t) = (-t)^{-1/4}\exp[^2/_3 i\,(-t)^{3/2} + {}^1/_4\pi i]$$
$$w'(t) = (-t)^{1/4}\exp[^2/_3 i\,(-t)^{3/2} - {}^1/_4\pi i] \qquad (t \ll 0)$$

The result obtained on some interval of values of t coincides with Expression (4.6).

For $t \gg 0$, i.e. in the region of the shadow for longitudinal displacements, the integral of (5.1) is calculated by means of the residues of the integrand at the zeros of the denominator. Here we obtain a series which when summed with the series (4.2) coincides with the series (3.2) representing the solution of the problem in the region of the shadow.

Thus, Expression (5.1) ties together the results obtained earlier for the solutions in the regions of shadow and light for the longitudinal displacements.

Analogously, we obtain for ω_2

$$\mathbf{w}_g{}^{sv} = \frac{e^{-1/4\pi i}\cos\varphi}{2\sqrt{\pi\sin\vartheta}}\left(\frac{k_2 a}{2}\right)^{-1/6}\frac{a}{r}\left\{\int_0^\infty \frac{p_2{}^{1/2}(\tau)\,[\overline{w}'(\tau) - q_2(\tau)\,\overline{w}(\tau)]}{V(\tau, r, 1)\,[w'(\tau) - q_2(\tau)\,w(\tau)]}\,e^{i\psi_2(\tau)}\,\times\right.$$

$$\times\left[\mathbf{r}_0\frac{a}{r}\,p_2(\tau) - \boldsymbol{\vartheta}_0 V^2(\tau, r, 1)\right]d\tau - 2e^{1/6\pi i}\int_0^\infty \frac{p_2{}^{1/2}(\tau e^{2/3\pi i})\,[v'(\tau) - q_2(\tau e^{2/3\pi i})\,e^{2/3\pi i}v(\tau)]}{V(\tau e^{2/3\pi i}, r, 1)\,[\overline{w}'(\tau) - q_2(\tau e^{2/3\pi i})\,e^{2/3\pi i}\overline{w}(\tau)]}\,\times$$

$$\left.\times\exp\left[i\psi_2(\tau e^{2/3\pi i})\right]\left[\mathbf{r}_0\frac{a}{r}\,p_2(\tau e^{2/3\pi i}) - \boldsymbol{\vartheta}_0 V^2(\tau e^{2/3\pi i}, r, 1)\right]d\tau\right\} \qquad (5.3)$$

$$\mathbf{w}_g{}^{sh} = \frac{e^{-3/4\pi i}\varphi_0\sin\varphi}{2\sqrt{\pi\sin\vartheta}}\left(\frac{k_2 a}{2}\right)^{-1/6}\frac{a}{r}\left\{\int_0^\infty \frac{p_2{}^{1/2}(\tau)\,L(\tau)}{V(\tau, r, 1)}\,e^{i\psi_2(\tau)}\,d\tau - 2e^{1/6\pi i}\,\times\right.$$

$$\left.\times\int_0^\infty \frac{p_2{}^{1/2}(\tau e^{2/3\pi i})\,M(\tau)}{V(\tau e^{2/3\pi i}, r, 1)}\exp\left[i\psi_2(\tau e^{2/3\pi i})\right]d\tau\right\} \qquad (5.4)$$

$$p_2(\tau) = 1 + \left(\frac{y}{2}\right)^{-2/3}\frac{\tau}{2}, \quad L^e(\tau) = \frac{\overline{w}'(\tau)}{w'(\tau)}, \quad L^r(\tau) = \frac{\overline{w}(\tau)}{w(\tau)}, \quad M^e(\tau) = \frac{v'(\tau)}{\overline{w}'(\tau)}$$

$$M^r(\tau) = \frac{v(\tau)}{\overline{w}(\tau)}, \quad q_2{}^e(\tau) = \frac{[2p_2{}^2(\tau) - 1]^2}{4p_2{}^2(\tau)\sqrt{p_2{}^2(\tau) - \varepsilon^2}}\left(\frac{k_2 a}{2}\right)^{1/3}, \quad q_2{}^r(\tau) = \frac{p_2{}^2(\tau)}{\sqrt{p_2{}^2(\tau) - \varepsilon^2}}\left(\frac{k_2 a}{2}\right)^{1/3}$$

$$\psi_2(\tau) = k_2 a\,p_2(\tau)\,[\vartheta - 1/2\pi - \cos^{-1} a p_2(\tau)/r] + k_2 r V^2(\tau, r, 1)$$

It is easy to show that Equations (5.3) and (5.4) connect the solutions in the region of the geometric shadow with those in the illuminated region.

In the region ω_3 we obtain, instead of (4.7),

$$\mathbf{w}_g{}^{sv} = \frac{e^{1/4\pi i}\cos\varphi}{2\sqrt{\pi\sin\vartheta}}\frac{a}{r}\left(\frac{k_1 a}{2}\right)^{-1/6}\left\{\int_0^\infty \frac{w'(t) + q_1(t)\,w(t)}{w'(t) - q_1(t)\,w(t)}\,\frac{p_1{}^{1/2}(t)\exp[i\psi_3(t)]}{V(t, r, \varepsilon)}\,\times\right.$$

$$\times\left[\mathbf{r}_0\varepsilon\frac{a}{r}\,p_1(t) - \boldsymbol{\vartheta}_0 V^2(t, r, \varepsilon)\right]dt - e^{2/3\pi i}\int_0^\infty \frac{\overline{w}'(t) + e^{2/3\pi i}q_1(te^{2/3\pi i})\,\overline{w}(t)}{\overline{w}'(t) - e^{2/3\pi i}q_1(te^{2/3\pi i})\,\overline{w}(t)}\,\times$$

$$\left.\times\frac{p_1{}^{1/2}(te^{2/3\pi i})}{V(te^{2/3\pi i}, r, \varepsilon)}\exp\left[i\psi_3(te^{2/3\pi i})\right]\left[\mathbf{r}_0\varepsilon\frac{a}{r}\,p_1(te^{2/3\pi i}) - \boldsymbol{\vartheta}_0 V^2(te^{2/3\pi i}, r, \varepsilon)\right]dt\right\} \qquad (5.5)$$

$$\psi_3(t) = k_1 a p_1(t)\,[\vartheta - 1/2\pi + 2\cos^{-1}\varepsilon p_1(t) - \cos^{-1}\varepsilon a p_1(t)/r] -$$
$$- 2k_2 a V^2(t, a, \varepsilon) + k_2 r V^2(t, r, \varepsilon)$$

These integrals may also be computed by numerical quadrature using the tables of [19]. the integrals of (5.5) provide a continuous transition from the solution representing the displacements of the SV waves in region (3) to the solution in region (4) (Fig.3).

In this way, for each region of Fig.3 the asymptotic expressions for the displacements $(k_1 a,\ k_2 a \gg 1)$ are represented in the form

(1) $\mathbf{w} = \mathbf{w}_1{}^p + \mathbf{w}_{11}{}^{sv} + \mathbf{w}_{12}{}^{sv} + \mathbf{w}_1{}^{sh}$

(2) $\mathbf{w} = \mathbf{w}_1{}^p + \mathbf{w}_{11}{}^{sv} + \mathbf{w}_{22}{}^{sv} + \mathbf{w}_2{}^{sh} + \mathbf{w}_g{}^{sv} + \mathbf{w}_g{}^{sh}$

(3) $\mathbf{w} = \mathbf{w}_2{}^p + \mathbf{w}_g{}^p + \mathbf{w}_{11}{}^{sv} + \mathbf{w}_{22}{}^{sv} + \mathbf{w}_2{}^{sh} + \mathbf{w}_g{}^{sv} + \mathbf{w}_g{}^{sh}$

(4) $\mathbf{w} = \mathbf{w}_2{}^p + \mathbf{w}_g{}^p + \mathbf{w}_{21}{}^{sv} + \mathbf{w}_{22}{}^{sv} + \mathbf{w}_2{}^{sh} + \mathbf{w}_g{}^{sv} + \mathbf{w}_g{}^{sh}$

Keller's geometric method in the theory of diffraction [25] can be extended to the case of diffraction of elastic waves by arbitrary bodies in the same way as was done in [17]. The diffraction coefficients and decay exponents (in terms of which the elastic displacements are expressed in Keller's method) can be taken from [17] for P and SV waves. For SH waves these can be obtained by comparison with the asymptotic solution obtained in the present paper.

The author is grateful to G.A. Skuridin for his interest in this work.

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Translated by A.R.R.